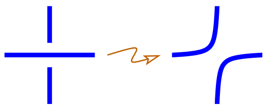
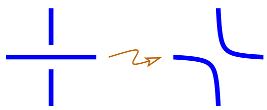
The Kauffman bracket and the Jones polynomial [Ka1]

Let L be a link diagram.

A-splitting: 

B-splitting: 

A *state* s is a choice of either A- or B-splitting at every classical crossing.

$\alpha(s) = \#(\text{of } A\text{-splittings in } s)$

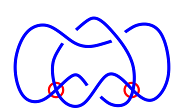
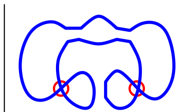
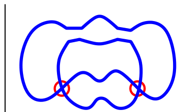
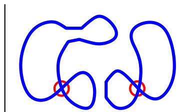
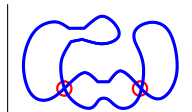
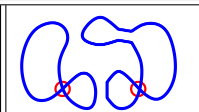
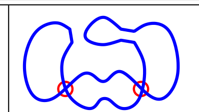
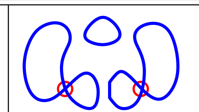
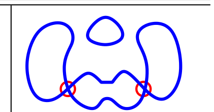
$\beta(s) = \#(\text{of } B\text{-splittings in } s)$

$\delta(s) = \#(\text{of circles in } s)$

$$[L](A,B,d) \;:=\; \sum_{s\in\mathcal{S}(L)} A^{\alpha(s)}\, B^{\beta(s)}\, d^{\delta(s)-1}$$

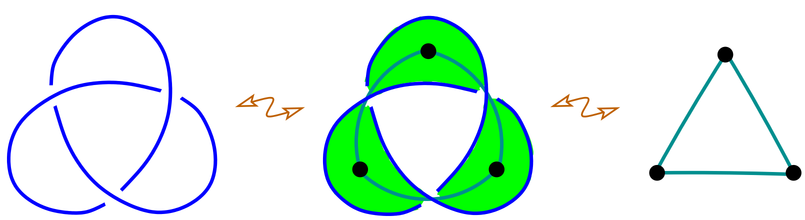
$$J_L(t) \;:= (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

Example

				
(α,β,δ)	$(3,0,1)$	$(2,1,2)$	$(2,1,2)$	$(1,2,1)$
				
	$(2,1,2)$	$(1,2,1)$	$(1,2,3)$	$(0,3,2)$

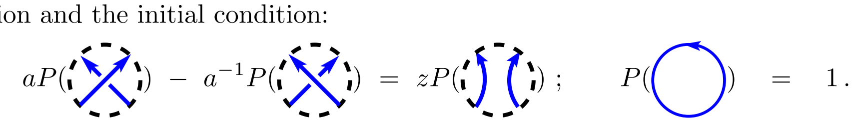
$$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d; \qquad J_L(t) = 1$$

Thistlethwaite's Theorem [Ka1] *Up to a sign and multiplication by a power of t the Jones polynomial $J_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_\Gamma(-t, -t^{-1})$.*



HOMFLYPT polynomial [Lik]

The *HOMFLYPT polynomial* $P(L) \in \mathbb{Z}(a^{\pm 1}, z^{\pm 1})$ is an unframed link invariant. It is defined as the Laurent polynomial in two variables a and z with integer coefficients satisfying the following skein relation and the initial condition:



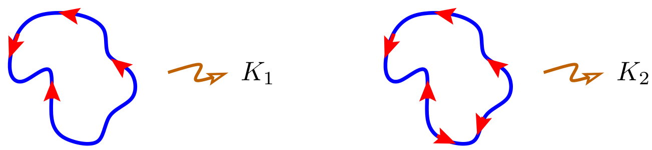
The existence of such an invariant is a difficult theorem [HOM, PT].

$$P(L)\Big|_{\substack{z=t^{1/2}-t^{-1/2}\\ a=t^{-1}}} = J_L(t)$$

Arrow polynomial [DK]

If a slitting does not respect the orientation, we put two arrows on the branches of the splitting oriented counterclockwise near the crossing:

Cancellation of arrows. All pairs of arrows which are next to each other on a state circle and pointing in the same direction are canceled.



With each state circle c we associate the variable $K_{i(c)}$ as follows. The $i(c)$ is equal to half of the number of arrows along c remained after the cancellations. We set $K_0 = 1$ and define the *arrow bracket polynomial* as

$$[L]_A(A, B, d, K_i) \ := \sum_{s\in\mathcal{S}(L)} A^{\alpha(s)}\, B^{\beta(s)}\, d^{\,\delta(s)-1} \prod_{c\in s} K_{i(c)} \ .$$

Thus the arrow bracket polynomial $[L]$ becomes a polynomial in infinitely many variables $A, B, d, K_1, K_2, \dots$. However, for a concrete diagram L only finitely many K's appear in it.

The standard substitution $B := A^{-1}$, $d := -A^2 - A^{-2}$ gives the *normalized Dye-Kauffman arrow polynomial* [DK]:

$$\langle L\rangle_{NA} := (-A^3)^{-w(L)}[L]_A(A, A^{-1}, -A^2 - A^{-2}, K_i) \ ,$$

which is an invariant of virtual links. The invariance under the Reidemeister moves follows from the cancellation rule. A remarkable observation of H. Dye and L. Kauffman is that for classical link diagrams, all arrows will be canceled out and the K's variables do not occur in the arrow polynomial. In this case it is essentially equivalent to the Jones polynomial (after the further substitution $A = t^{-1/4}$).

Arrow Thistlethwaite's theorem is proved in [BBC].

References

[BBC] R. Bradford, C. Butler, S. Chmutov, *Arrow ribbon graphs*, Journal of Knot Theory and its Ramifications, **21**(13) (2012) # 1240002 (16 pages).

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[HOM] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985) 239–246.

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